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Solving an inverse heat conduction problem using a non-integer identified model

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Abstract

An inverse heat conduction problem in a system is solved using a non-integer identified model as the direct model for the estimation procedure. This method is efficient when some governing parameters of the heat transfer equations, such as thermal conductivity or thermal resistance, are not known precisely. Reliability of the inversion depends on the precision of the identified model. From considerations on the analytical solutions in simple cases and on the definition of non-integer (or fractional) derivative, the non-integer model appears to be the most adapted. However, some experiments do need to be carried out on the physical thermal system before it can be identified. An application that consists in estimating the heat flux in a turning tool insert during machining is presented. First, identification is performed using a specific apparatus that permits a simultaneous measurement of temperature and heat flux in the insert. Then, during machining, heat flux can be estimated from temperature using this identified model. © 2001 Elsevier Science Ltd. All rights reserved.

Keywords: Inverse problem; System identification; Fractional derivative; Non-integer model; Turning process; Tool; Heat flux

1. Introduction

Let us consider a system Ω, eventually constituted from several sub-domains $\partial\Omega$ and submitted to a heat flux $\phi(t)$ on a part Γ of its external surface (Fig. 1). Solving the inverse heat conduction problem consists in estimating the heat flux $\phi(t)$ from temperature measurements in the system. As we consider a single heat flux, only the temperature $T(P,t)$ in the body at point P along the time is necessary. Obviously, reliability of the estimation of $\phi(t)$ is better considering the smallest distance between the point P and the external surface Γ, where the heat flux is applied.

A model, that expressed the heat flux $\phi(t)$ according to the temperature $T(P,t)$ along the time, is also necessary in the procedure for the resolution of the inverse problem. This model is classically issued from the governing equations of transient heat transfer and associated boundary conditions. According to the complexity of the system, several parameters need to be known in order to solve these equations. These can be the thermal conductivity or the specific heat of materials that enter into the constitution of the sub-domains. On the other hand, thermal resistances between the sub-domains are much more difficult to estimate than thermophysical properties.

In face of such difficulties, another approach can be considered. It consists in identifying the relationship between the heat flux $\phi(t)$ and the temperature $T(P,t)$ by applying a known heat flux, like a step for example, and measuring the temperature at point P. The most interesting model is the impulse response since it can be directly used in the procedure of inversion. Unfortunately,

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Nomenclature

$a_i,\ b_j$	coefficients of the non-integer model
A_l	modal coefficient
a_{p}	depth of cut (mm)
C_{p}	specific heat (J kg^{-1} K^{-1})
e	output error
f	feed rate (mm tr^{-1})
$F(s)$	Laplace transfer function
h	sampling period
$I(t)$	impulse response
J	quadratic criterion
$\mathbf{J}'_{\boldsymbol{\theta}}$	gradient
$\mathbf{J}''_{\boldsymbol{\theta\theta}}$	hessian
M	number of data
n	common real order of the non-integer model
$n_{a_i},\ n_{b_j}$	differentiation orders of the non-integer model
q_i	heat source in region i
r	number of future time steps
s	Laplace variable
$\mathbf{S}(t, A_l)$	output sensitivity function with respect to A_l
t	time
$T(P,t)$	temperature at point P and time t
$\overline{T}(P,s)$	Laplace transform of $T(P,t)$
$\hat{T}(Kh, \hat{\boldsymbol{\theta}})$	calculated temperature
V_{c}	cutting speed (m s^{-1})
Y	measured temperature at point P
z_l	roots of the polynomial function $N(s^n) = 0$

Greek symbols

α	thermal diffusivity (m^2 s^{-1})
$\alpha_i,\ \beta_j$	coefficients of the identification model
$\phi(t)$	heat flux
$\overline{\phi}(s)$	Laplace transform of $\phi(t)$
Γ	part of external surface of Ω
λ	thermal conductivity (W m^{-1} K^{-1})
λ_l	eigenvalues of the non-integer model (relation (7))
Ω	physical system
$\partial\Omega$	sub-domain of Ω
$\boldsymbol{\theta}$	vector of unknown parameters
$\hat{\boldsymbol{\theta}}$	vector of estimated parameters
$\boldsymbol{\theta}_{\mathrm{opt}}$	optimal value of $\hat{\boldsymbol{\theta}}$
ρ	density (kg m^{-3})
ξ	Marquardt parameter

Subscripts

$i,\ j,\ k,\ l$	iteration subscripts
j	imaginary unit $(\sqrt{(-1)})$

Other symbols

$\mathrm{D} = \mathrm{d}/\mathrm{d}t$	differential operator
E	estimator
$\mathbf{L}^{-1}$	inverse Laplace transform operator
$*$	convolution product

this kind of response is generally difficult to obtain for two reasons. The first concerns the difficulty in generating and measuring a homogeneous pulse over the surface Γ. The second refers to the reproducibility of this response, this means that a great dispersion can be found in the results considering successive trials.

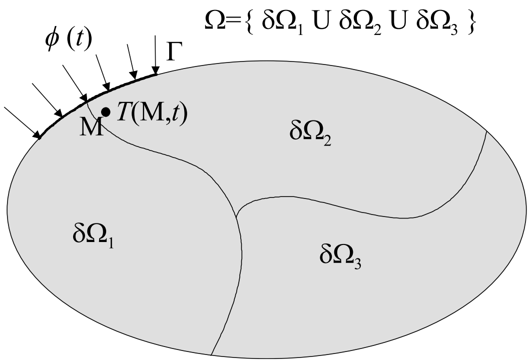

Fig. 1. System constituted by sub-domains $\delta\Omega_i$. Temperature $T(M,t)$ at point M is measured in order to estimate the unknown heat flux $\phi(t)$ along the time.

For these reasons, it is preferable to excite the system with a measurable random heat flux. Such a signal ensures redundant measurements especially for the small time where computation of the impulse response requires the most important precision. Assuming linear heat transfer in the system, the lagging effect leads to naturally express the model under the following discrete time form:

$$\alpha_0 T(P, Kh) + \alpha_1 T(P, (K-1)h) + \cdots + \alpha_I T(P, (K-I)h) = \beta_0 \phi(Kh) + \beta_1 \phi((K-1)h) + \cdots + \beta_J \phi((K-J)h), \quad (1)$$

where the continuous time has been decomposed into $(M-1)$ intervals, h being the sampling period. Then, $T(P, Kh)$ and $\phi(Kh)$ are, respectively, the temperature at point P and the heat flux at time $t = Kh$ $(0 \leqslant K \leqslant M)$. Parameters to be estimated are α_i $(0 \leqslant i \leqslant I)$ and β_j $(0 \leqslant j \leqslant J)$. The continuous time representation of such a relationship is

$$a_0 \mathrm{D}^{(n_{a_0})} T(P,t) + \cdots + a_I \mathrm{D}^{(n_{a_I})} T(P,t) = b_0 \mathrm{D}^{(n_{b_0})} \phi(t) + \cdots + b_J \mathrm{D}^{(n_{b_J})} \phi(t). \quad (2)$$

The differentiation orders are $(n_{a_0}, \ldots, n_{a_I}) = (0, 1, \ldots, I)$ and $(n_{b_0}, \ldots, n_{b_J}) = (0, 1, \ldots, J)$, and D denotes the differential operator $\{\mathrm{d}/\mathrm{d}t\}$. If one considers a zero initial temperature at point P, the Laplace transform of relation (2) leads to the transfer function $F(s)$ as

$$\frac{\overline{T}(P,s)}{\overline{\phi}(s)} = F(s) = \frac{b_0 + b_1 s + \cdots + b_J s^J}{a_0 + a_1 s + \cdots + a_I s^I} \tag{3}$$

with $\overline{T}(P,s) = \int_0^\infty T(P,t)\mathrm{e}^{-st}\,\mathrm{d}t$ and $\overline{\phi}(s) = \int_0^\infty \phi(t)\mathrm{e}^{-st}\,\mathrm{d}t$.

From recent works [1], we have shown that a more appropriate form of the relationship is obtained considering real (integer or non-integer) differentiation orders $(n_{a_0}, \ldots, n_{a_I}, n_{b_0}, \ldots, n_{b_J})$ in relation (2). The fractional derivative, introduced by Riemann and Liouville in 1831, is a generalization of the classical definition of the derivative [2,3]. It has been recently used in several applications in the domain of automatic (robust control an identification [4,5]). Thus, the model expressed by the relation (2) is often called a non-integer model.

Oldham and Spanier have proved that the Laplace transform for fractional derivative is a generalization of the usual one applied for integer derivative. Then, relation (3) is still valid and exponents of the Laplace variable s are then real.

Let us consider a very simple configuration where the system is homogeneous (thermal conductivity λ, density ρ, specific heat C_p) semi-infinite and point P is located on the surface Γ. The expression of the transfer function is [1]

$$F(s) = \frac{\overline{T}(0,s)}{\overline{\phi}(s)} = \frac{1}{\sqrt{\lambda\rho C_\mathrm{p}}} s^{-1/2}. \tag{4}$$

In the continuous time-domain this relation corresponds to

$$\mathrm{D}^{1/2} T(0,t) = \frac{1}{\sqrt{\lambda\rho C_\mathrm{p}}} \phi(t). \tag{5}$$

It is clear that the derivative of $T(0,t)$ is of real order and equal to 0.5. If one considers the point P at the abscise x inside the medium, a similar result is obtained, using series expansion technique, as

$$\frac{\overline{T}(P,s)}{\overline{\phi}(s)} = \frac{1}{\sqrt{\lambda\rho C_\mathrm{p}}} \exp\left(-\sqrt{\frac{s}{\alpha}}x\right) = \frac{1}{\sqrt{\lambda\rho C_\mathrm{p}}} \sum_{n=0}^{\infty} \frac{(-1)^n s^{(n-1)/2} x^n}{\alpha^n n!}. \tag{6}$$

In this case, the exponents of s are multiples of the real value 0.5. Oldham and Spanier [6–8] have demonstrated that such a real value is obtained considering semi-infinite planar, cylindrical and spherical geometric configurations. On the other hand, it does not change applying a prescribed temperature on the surface of the medium instead of a prescribed heat flux. In the case of finite domains, the series expansion technique allows the transfer function to be expressed as in relation (6) where the exponents of s are multiples of 0.5. Obviously, this result is only valid in the field of heat transfer by diffusion.

In this paper, the estimation method of the non-integer model parameters is presented. The identified model is then used to compute the impulse response of the system in order to solve the inverse heat conduction problem. The procedure of inversion is based on the classical sequential function specification method developed by Beck et al. [9].

An application is treated that consists in estimating the heat flux in a tool during machining by turning. First, a specific experiment that permits to apply a known heat flux at the tip of the tool is realized. A non-integer model is achieved from these measurements and the impulse response is computed. Then, the heat flux is estimated during real machining. A constant heat flux functional form is used in the procedure of sequential estimation.

2. System identification

2.1. Presentation and numerical calculation

In the model described by relation (2), real values of the differentiation orders are considered. The model is used to express the temperature in the system according to the heat flux. A common real order n can be determined such that all differentiation orders $(n_{a_0}, \ldots, n_{a_I}, n_{b_0}, \ldots, n_{b_J})$ can be expressed as the product of n by an integer. The transfer function corresponding to the differential equation (2) is thus

$$\frac{\overline{T}(P,s)}{\overline{\phi}(s)} = \frac{b_{N-1}s^{(N-1)n} + \cdots + b_0}{s^{Nn} + a_{N-1}s^{(N-1)n} + \cdots + a_0} = \frac{N(s^n)}{D(s^n)}, \tag{7}$$

where n is the common real differentiation order and $N(s^n)$ and $D(s^n)$ are the two polynomial functions of the s^n variable.

The identification model used here results from a breakdown of the previous non-integer rational transfer function into partial fractions. Under this form, called *developed modal form*, the temperature is a linear combination of several modes, (called eigenmodes in the field of automatic control). Each mode is characterized by three parameters: the modal coefficient, the eigenvalue and the differentiation order, and provides its own specific dynamic contribution to the response of the system. The developed modal form of the relation (7) is expressed therefore as

$$\overline{T}(P,s) = \sum_{l=1}^{N} \frac{A_l}{s^n - \lambda_l} \overline{\phi}(s) = \sum_{l=1}^{N} \overline{T}_l(s). \tag{8}$$

The corresponding factorized modal form is thus

$$\overline{T}(P,s) = \frac{A\prod_{l=1}^{N}(s^n - z_l)}{\prod_{l=1}^{N}(s^n - \lambda_l)}\overline{\phi}(s), \tag{9}$$

where z_l are the roots of $N(s^n) = 0$.

The aim of identification is to estimate vector $\boldsymbol{\theta}$ defined by the model parameters: $\boldsymbol{\theta}^{\mathrm{T}} = [n, A_1, \lambda_1, \ldots A_N, \lambda_N]$ [10].

In practice, the fractional derivatives are replaced by their discrete approximation, considering a constant sampling period h (see Appendix A for the demonstration), namely

$$\mathrm{D}^n f(Kh) = \frac{1}{h^n}\sum_{k=0}^{K}(-1)^k\binom{n}{k}f((K-k)h) \tag{10}$$

with

$$\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k!}, \quad n \in \mathrm{R}. \tag{11}$$

Then, considering an estimate $\hat{\boldsymbol{\theta}}$ of $\hat{\boldsymbol{\theta}}$, the calculated temperature $\hat{T}(Kh, \hat{\boldsymbol{\theta}})$ at point P and at time $t = Kh$, is computed from the following recursive equation system:

$$\begin{aligned} s^n - \lambda_1\overline{T}_1(s) = A_1\overline{\phi}(s) \Longleftrightarrow \sum_{k=0}^{K}\frac{(-1)^k}{h^n}\binom{n}{k}T_1((K-k)h) \\ - \lambda_1 T_1(Kh) = A_1\phi(Kh) \\ \vdots \\ s^n - \lambda_N\overline{T}_N(s) = A_N\overline{\phi}(s) \Longleftrightarrow \sum_{k=0}^{K}\frac{(-1)^k}{h^n}\binom{n}{k}T_N((K-k)h) \\ - \lambda_N T_N(Kh) = A_N\phi(Kh) \end{aligned} \tag{12}$$

and

$$\hat{T}(Kh, \hat{\boldsymbol{\theta}}) = \sum_{l=1}^{N} T_l(Kh). \tag{13}$$

2.2. Model order reduction

The non-integer model can be expressed under three equivalent forms: the rational transfer function in s^n (relation (7)), providing the various differentiation orders; the developed modal form (relation (8)), providing the eigenvalues; and the factorized modal form (relation (9)).

A critical choice in the identification procedure concerns the number N of eigenmodes in relation (8). For this, the modelization error, which is defined by the norm of the gap between the responses of the identified model and the real system, is introduced. Starting from a specific number N, this error does not significantly get decreased when increasing N. It comes from the fact that the response of the real system contains measurement noise and consequently leads to a value of z_l close to an eigenvalue λ_l within their confidence intervals. Thereby, this eigenmode can be rejected and a model reduction is obtained.

2.3. Model parameter estimation

2.3.1. Output error identification algorithm

The sampled data set is composed of M data pairs $[\phi(Kh), Y(Kh)]$ ($1 \leqslant K \leqslant M$ and h being the sampling period) where Y denotes the measured temperature at point P. The output error is given by

$$e(Kh, \hat{\boldsymbol{\theta}}) = Y(Kh) - \hat{T}(Kh, \hat{\boldsymbol{\theta}}). \tag{14}$$

Now $\boldsymbol{\theta}_{\mathrm{opt}}$, the optimal value of $\hat{\boldsymbol{\theta}}$, is obtained by minimization of the quadratic criterion

$$J(\hat{\boldsymbol{\theta}}) = \sum_{K=1}^{M} \mathrm{e}^2(Kh, \hat{\boldsymbol{\theta}}). \tag{15}$$

As output prediction $\hat{T}(Kh, \hat{\boldsymbol{\theta}})$ is non-linear with respect to $\hat{\boldsymbol{\theta}}$, a non-linear programming technique, in this case the Marquardt algorithm [11], is used to estimate $\hat{\boldsymbol{\theta}}$ iteratively

$$\boldsymbol{\theta}_{i+1} = \boldsymbol{\theta}_i - \left\{\left[\mathbf{J}''_{\boldsymbol{\theta\theta}} + \xi\mathbf{I}\right]^{-1}\mathbf{J}'_{\boldsymbol{\theta}}\right\}_{\hat{\boldsymbol{\theta}}=\boldsymbol{\theta}_i} \tag{16}$$

with

$\mathbf{J}'_{\boldsymbol{\theta}} = -2\sum_{K=1}^{M} e(Kh)\mathbf{S}(Kh, \hat{\boldsymbol{\theta}})$	gradient
$\mathbf{J}''_{\boldsymbol{\theta\theta}} \approx 2\sum_{K=1}^{M} \mathbf{S}(Kh, \hat{\boldsymbol{\theta}})\mathbf{S}^{\mathrm{T}}(Kh, \hat{\boldsymbol{\theta}})$	hessian
$\mathbf{S}(Kh, \hat{\boldsymbol{\theta}}) = (\partial\hat{T}(Kh, \hat{\boldsymbol{\theta}}))/\partial\boldsymbol{\theta}$	output sensitivity function
ξ	Marquardt parameter

This algorithm, often used in non-linear optimization, ensures robust convergence, even when $\hat{\boldsymbol{\theta}}$ is initialized at a value far from the true value.

2.3.2. Output sensitivity computation

The identification model structure makes output sensitivity computation easier. By partial differentiation of output prediction, we obtain

$$\begin{aligned} S(t, A_l) &= \frac{\partial\hat{T}(t, \hat{\boldsymbol{\theta}})}{\partial A_l} = \mathbf{L}^{-1}\left(\frac{1}{s^n - \lambda_l}\right) * \phi(t), \\ S(t, \lambda_l) &= \frac{\partial\hat{T}(t, \hat{\boldsymbol{\theta}})}{\partial\lambda_l} = \mathbf{L}^{-1}\left(\frac{A_l}{(s^n - \lambda_l)^2}\right) * \phi(t), \\ S(t, n) &= \frac{\partial\hat{T}(t, \hat{\boldsymbol{\theta}})}{\partial n} = \mathbf{L}^{-1}\left(\sum_{l=1}^{N} -\frac{A_l s^n \ln(s)}{(s^n - \lambda_l)^2}\right) * \phi(t), \end{aligned} \tag{17}$$

where * and $\mathbf{L}^{-1}$, respectively, denote the convolution product and the inverse Laplace transform.

Using the non-integer derivative discrete approximation (10), $(\partial\hat{T}(t,\hat{\boldsymbol{\theta}}))/\partial A_l$ and $(\partial\hat{T}(t,\hat{\boldsymbol{\theta}}))/\partial\lambda_l$ are easy to compute. Due to the $\ln(s)$ term, computation of the $(\partial\hat{T}(t,\hat{\boldsymbol{\theta}}))/\partial n$ is more complicated and is given in Appendix B.

The three sensitivity functions are linearly independent. This means that the identification of $\hat{\boldsymbol{\theta}}$ can be performed efficiently.

2.3.3. Confidence domain of the estimated parameters

Under the classical assumptions of zero mean and constant variance σ^2 of the prediction error [12,13], the estimated parameter covariance matrix is given by

$$\mathrm{cov}(\boldsymbol{\theta}_{\mathrm{opt}}) = \sigma^2\left(\sum_{K=1}^{M}\mathbf{S}(Kh,\boldsymbol{\theta}_{\mathrm{opt}})\mathbf{S}^{\mathrm{T}}(Kh,\boldsymbol{\theta}_{\mathrm{opt}})\right)^{-1}. \tag{18}$$

Using this matrix, the parameter variances (on the matrix diagonal) and the correlation coefficients between parameters can be easily computed.

A classical estimator of σ^2 is given by

$$E(\sigma^2) = \frac{\sum_{K=1}^{M}\mathrm{e}^2(Kh)}{M-2N-1}. \tag{19}$$

3. Resolution of the inverse heat conduction problem

As in the previous chapter, Y denotes the temperature measured during machining at point P. Let us compute the impulse response $I(Kh)(K=1,\dots,M)$ from the identified model with the optimal parameters $\boldsymbol{\theta}_{\mathrm{opt}}$. Using initial temperature $T_0(P)$ at point P, and the convolution theorem

$$T(P,Kh) = \sum_{l=0}^{K} I((K-l+1)h)\phi(Kh) + T_0(P), \quad 1\leqslant K\leqslant M. \tag{20}$$

We assume a constant heat flux functional form from K to $(K+r-1)$, where r is the number of future time steps, usually chosen to be about 3 or 4. Then substituting

$$\phi(Kh) = \phi((K+1)h) = \cdots = \phi((K+r-1)h) \tag{21}$$

in relation (20)

$$\begin{aligned} T(P,Kh) &= dh_1\phi(Kh) + \tilde{T}(P,Kh) \\ T(P,(K+1)h) &= dh_2\phi(Kh) + \tilde{T}(P,(K+1)h) \\ &\vdots \\ T(P,(K+r-1)h) &= dh_r\phi(Kh) + \tilde{T}(P,(K+r-1)h) \end{aligned} \tag{22}$$

with

$$\mathrm{d}h_j = \sum_{i=1}^{j} I(ih) \tag{23}$$

and

$$\tilde{T}(P,(K+j-1)h) = \sum_{l=1}^{K-1} I((K+j-l)h)\hat{\phi}(lh) + T_0(P), \quad 1\leqslant j\leqslant r. \tag{24}$$

The least-squares procedure for estimating $\phi(Kh)$, with the temperature measurements $Y(Kh), Y((K+1)h),\dots, Y((K+r-1)h)$, minimizes

$$S = \sum_{j=1}^{r}(Y((K+j-1)h) - T(P,(K+j-1)h))^2, \tag{25}$$

which is differentiated with respect to $\phi(Kh)$. Now $\partial S/(\partial\phi(Kh))$ is set equal to zero, and $\phi(Kh)$ is replaced by the estimate, providing

$$\hat{\phi}(Kh) = \frac{\sum_{j=1}^{r}(Y((K+j-1)h) - \tilde{T}(P,(K+j-1)h))\mathrm{d}h_j}{\sum_{j=1}^{r}\mathrm{d}h_j^2}. \tag{26}$$

The sequential procedure for estimation consists first in calculating $\mathrm{d}h_j$ and $\tilde{T}(P,(K+j+1)h)$ for $1\leqslant j\leqslant r$ from relations (23) and (24). The heat flux at time $t=Kh$ is then estimated from relation (25). Finally K is increased by one and the procedure is repeated till $K=M$.

4. Application

In turning (see Fig. 2), tearing of the material from the workpiece as a chip comes from the cutting between the tool insert and the cylindrical.

Assuming the tool insert is not affected by the cutting process, three specific regions appear to be concerned with the heat generation in the material cut as it is described in Fig. 3 (see also [14,15]). Heat source q_1 in region 1, known as the primary shear zone, comes from plastic deformation and viscous dissipation. Heat source q_2 in region 2, known as the secondary shear zone, is located at the tool-chip interface where the heated chip slides on the insert top surface. Finally, the last region, denoted as region 3, refers to the tool-workpiece interface where frictional rubbing of the workpiece on the tool insert flank leads to the heat source q_3. This last heat source exclusively depends on the tool wear. Each of these three heat sources contributes to the heat flux through the tool denoted $\phi(t)$.

We now use our method to estimate the heat flux $\phi(t)$ in the tool during real machining.

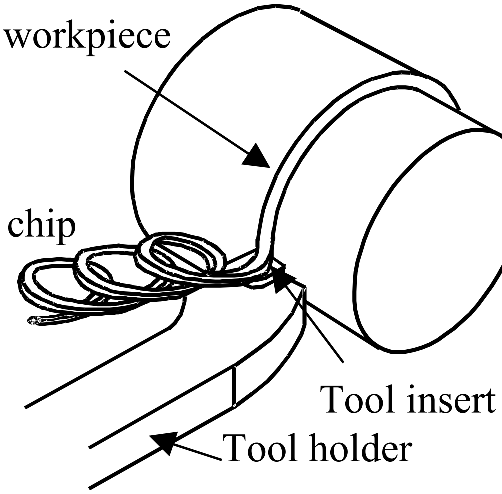

Fig. 2. Schematic representation of a turning process. The three parts of the system are the tool (tool insert and tool holder), the chip and the workpiece.

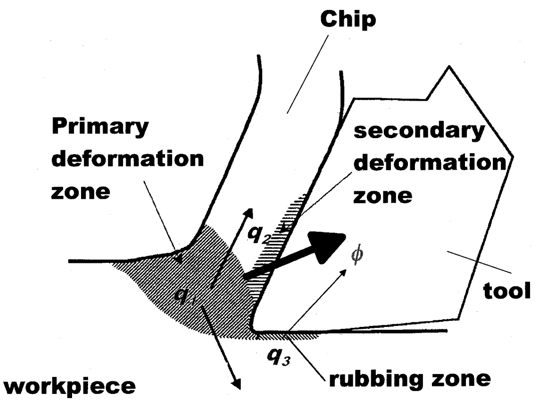

Fig. 3. Heat transfer in a cutting process. Three heat sources are q_1, q_2 and q_3 that contribute to the heat fluxes ϕ_w, ϕ_c and ϕ_t, respectively, in the workpiece, the chip and the tool.

4.1. Material and experimental design

For the model identification stage, the user needs to measure simultaneously: the temperature $Y(t)$ using a thermocouple embedded close to the tip of the insert, and the heat flux $\phi(t)$ through the tool. However, heat flux $\phi(t)$ cannot be measured during the turning process. A specific experiment has been designed which permits the heat flow to be measured (Fig. 4). A constantan wire is wound around the tip of the insert and serves as a heat resistor. The heat flux $\phi(t)$ is equal to the electric power provided to the resistor. The resistor is covered by an insulating material to make sure that the total heat flux generated by the resistor goes through the tool. The temperature $Y(t)$ is measured using a thermocouple

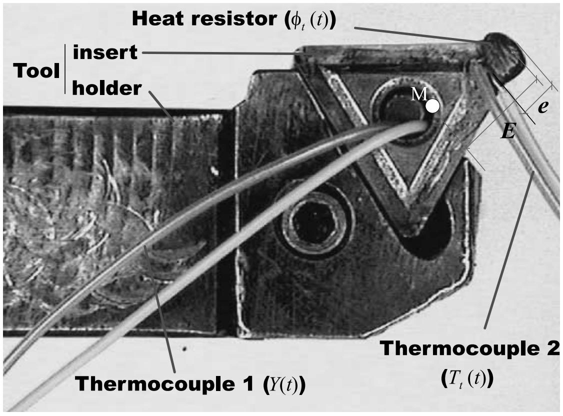

Fig. 4. Tool with embedded thermocouples and heat resistor simulating the thermal behavior at the tip of the insert in a cutting process. The length of the tool insert covered by the winding is $e = 3$ mm from the tip. Distance of the thermocouple from the tip of the insert is $E = 8$ mm.

placed under the insert mounting screw at 0.8 cm from the tip of the insert.

A SANDVIK COROMANT TNMG 16 04 08-23 uncoated carbide insert is used. A type K (chromel–alumel) thermocouple with a sensitivity of 40 μV/°C is used with a compensator for fluctuation in ambient temperature.

The electrical power is provided by a stabilized electrical supply. A Vishay conditioning amplifier (gain 3000) amplifies the signals from the thermocouple. Finally, a Nicolet type 310 is used for the data acquisition of $\phi(t)$ and $Y(t)$.

4.2. Model parameter estimation

The apparatus is now used to identify the thermal system. Since variances of the parameter estimates depend on the covariance function of the input signal, a pseudo-random binary signal (PRBS) input current is provided. This permits an efficient fitting of the model $\hat{T}(t, \hat{\boldsymbol{\theta}} = \boldsymbol{\theta}_{opt})$ with the experimental temperature values $Y(t)$ for a wide-frequency band. It must be stressed that the identified model is not necessarily valid if the turning process duration is longer than that of the identification experiment. This means that the duration of the input signal at the identification stage must be longer than the expected turning process duration.

A 2000 input–output data set was sampled at $h = 0.1$ s. The sampling period must be chosen quite short for reproduction of the dynamic behavior of the system to be faithful even at high frequencies. The identification model used is composed of three eigenmodes and therefore seven parameters. Table 1 gives estimation results and Fig. 5 illustrates the identification performance. The temperature predicted by the non-

Table 1
Estimated parameters (A_i, λ_i, n) along with their standard deviations ($\sigma(A_i)$, $\sigma(\lambda_i)$, $\sigma(n)$) and corresponding parameters (a_i, b_i, n_{a_i}, n_{b_i}) in the rational form of the continuous model

Mode 1	Mode 2	Mode 3	Order
$\left\{\begin{matrix} A_1 = -0.8323 - j0.6839 \\ \lambda_1 = -0.06364 + j0.3379 \end{matrix}\right\}$	$\left\{\begin{matrix} A_2 = -0.8323 + j0.6839 \\ \lambda_2 = -0.06364 - j0.3379 \end{matrix}\right\}$	$\left\{\begin{matrix} A_3 = 1.7041 \\ \lambda_3 = -0.48637 \end{matrix}\right\}$	$n = 0.5463$
$\left\{\begin{matrix} \sigma(A_1) = 0.0033 + j0.0144 \\ \sigma(\lambda_1) = 0.0035 + j0.0025 \end{matrix}\right\}$	$\left\{\begin{matrix} \sigma(A_1) = 0.0033 + j0.0144 \\ \sigma(\lambda_1) = 0.0035 + j0.0025 \end{matrix}\right\}$	$\left\{\begin{matrix} \sigma(A_3) = 0.0065 \\ \sigma(\lambda_3) = 0.0068 \end{matrix}\right\}$	$\sigma(n) = 0.0027$
$\left\{\begin{matrix} a_0 = 0.0575 \\ n_{a_0} = 0 \end{matrix}\right\}, \left\{\begin{matrix} a_1 = 0.1801 \\ n_{a_1} = 0.546 \end{matrix}\right\}, \left\{\begin{matrix} a_2 = 0.613 \\ n_{a_2} = 1.0927 \end{matrix}\right\}, \left\{\begin{matrix} a_3 = 1 \\ n_{a_3} = 1.639 \end{matrix}\right\}$			
$\left\{\begin{matrix} b_0 = 0.374 \\ n_{b_0} = 0 \end{matrix}\right\}, \left\{\begin{matrix} b_1 = -0.236 \\ n_{b_1} = 0.546 \end{matrix}\right\}, \left\{\begin{matrix} b_2 = 0.0393 \\ n_{b_2} = 1.0927 \end{matrix}\right\}$			

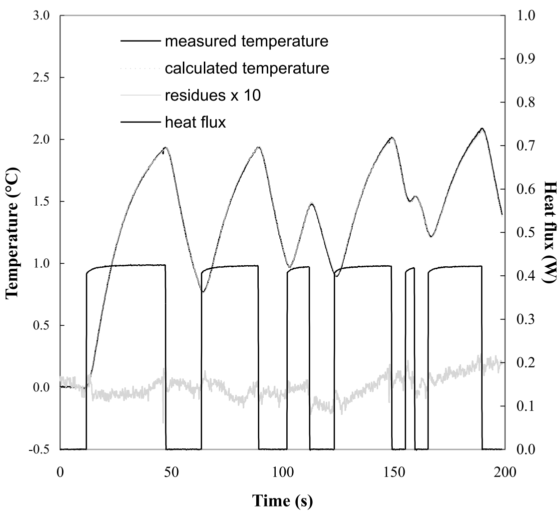

Fig. 5. Temperature measurements $Y(t)$ and calculated temperatures $T(t)$ for a wave sequence of $\phi(t)$, residues are denoted as $e(t)$.

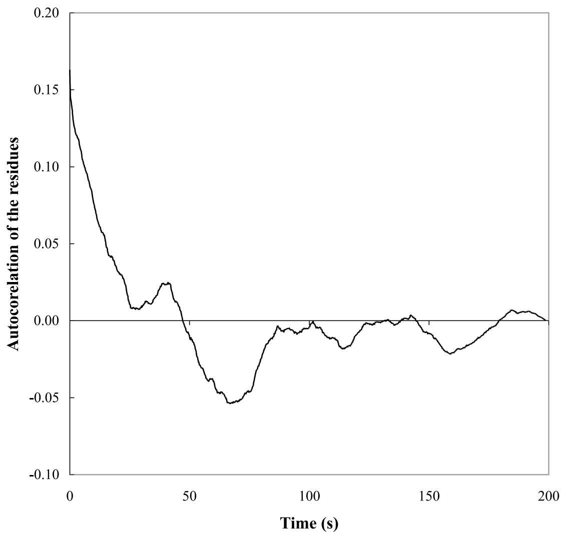

Fig. 6. Autocorrelation function of the residues $e(t)$ for validation of white noise measurements.

integer model is almost identical to the real temperature and the two traces cannot be seen as separate.

As it was mentioned previously, confidence domains of the identified parameters are calculated assuming a zero mean output error. In order to justify this assumption, the autocorrelation function of the residues is plotted in Fig. 6. As one can see, the mean of the autocorrelation function is close to zero but a bias occurs. This bias comes from the modelization error that grows when the number of parameters in relation (8) decreases. Thereby, the residues are not only constituted by the measurement errors but also by the modelization error.

When more than three eigenmodes were chosen, the model reduction brought the number back to three. This means that the modelization error cannot yet be significantly minimized. As it can be viewed in Fig. 7, the sensitivity functions for each parameter are of the same magnitude. This result permits the confidence domain of each estimated parameter to be of the same magnitude.

In order to validate the identified model, we repeat the experiment replacing the random input signal by a step of heat flux (Fig. 8). As one can see, the calculated temperature is very close to the experimental results. This confirms the reliability of the identified model and the weak modelization bias.

4.3. Estimation of $\phi(t)$ in a cutting process

The previous identified non-integer model is then used in the procedure for the estimation of the heat flux

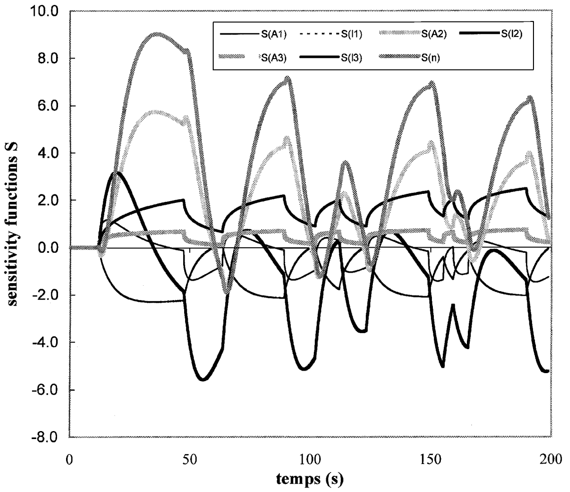

Fig. 7. Sensitivity functions of the estimated parameters calculated from their analytical expressions.

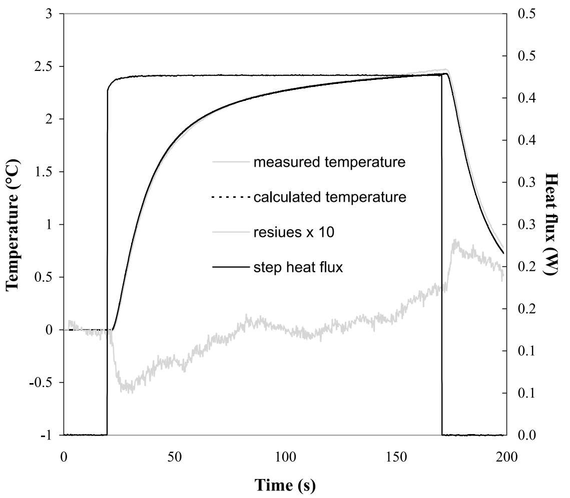

Fig. 8. Validation of the identified model on the experimental step response of the system.

during a turning process. The turning parameters are: cutting speed $V_c = 90$ m/s, feed rate $f = 0.3$ mm/tr and depth of cut $a_p = 2$ mm. The workpiece is made of steel 42CD4, 35 mm is the diameter before cutting.

The impulse response is first computed using the previous identified model. The estimated heat flux and experimental values of the temperature $Y(t)$ are represented in Fig. 9. The sampling period is $h = 0.5$ s and $M = 160$ is the number of data. The number of future time steps has been taken as equal to $r = 4$.

The computation times are less than 2 s on a PC Pentium III 450 MHz.

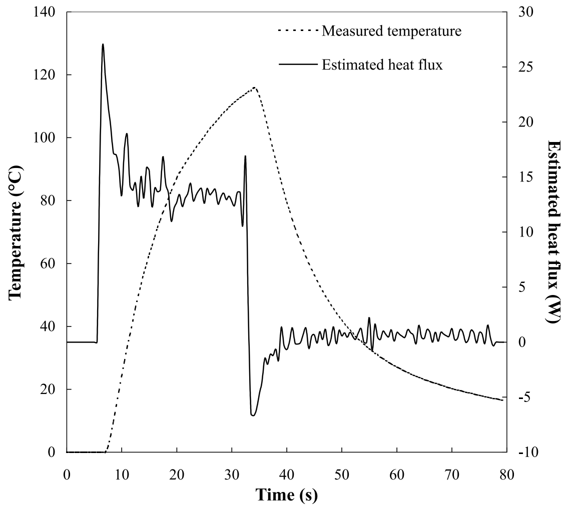

Fig. 9. Estimation of the heat flux in the tool during machining from the non-integer identified model. Number of future time steps is $r = 4$.

5. Conclusions

The use of a non-integer identified model as the direct model for the resolution of the inverse problem offers some specific advantages. First, the knowledge of thermophysical properties, such as thermal conductivity or specific heat, is not required. Likewise, those concerning thermal resistance and stationary boundary conditions that have no interest for the user. Secondly, the identification method does not depend on the spatial diffusion of heat in the medium. In other words, one-dimensional or three-dimensional diffusion requires the same computational time for the model identification.

On the other hand, it has been viewed in the introduction that considering a real differentiation order is in physical agreement with the analytical solutions found in various geometric configurations. Furthermore, all the differentiation orders are multiples of a single one. This result leads to a meaningful reduction in the number of unknown parameters in the expression of the transfer function in the form of relation (7). Concerning the resolution of the inverse problem from the identified model, relation (10) shows that the fractional derivative of a function at time t involves all the past of this function till the initial time $t = 0$. The inversion procedure takes a great benefit of this fact.

The principle weakness of this approach is that the estimated flux and temperature depend in the first place on the reliability of the identified models. Thereby, a great precision had to be put on the experimental laboratory design at the stage of model identification. On the other hand, the duration of the process cannot exceed the duration time for the model identification.

Finally, concerning the application, the heat flux in a tool, $\phi(t)$ has been estimated from temperature measurements, $T(P,t)$, in an interior point of the insert tool during machining by turning. The direct model that expressed $\phi(t)$ according to $T(P,t)$ has been identified from a specific apparatus. The inverse problem has been solved using the sequential method. The computation times for the estimation of the heat flux in the tool are very small compared to the duration of the process. Thereby, as it was envisaged by Oxley [16], the model can be used in the application of an adaptive control of the cutting temperature.

Appendix A. Fractional derivative of the function f (t)

By definition the derivative of the function $f(t)$ is

$$\mathrm{D}^1 f(t) = \lim_{h\to 0} \frac{f(t) - f(t-h)}{h}.$$

Using a sampling interval h of the time t, i.e., $t = Kh$, lead to

$$\mathrm{D}^1 f(t) = \frac{f(Kh) - f((K-1)h)}{h}.$$

Introducing the operator q defined by $q^{-1} f(Kh) = f((K-1)h)$ one obtains

$$\mathrm{D}^1 f(t) = \frac{1-q^{-1}}{h} f(Kh).$$

The same calculus is achieved at the order 2

$$\mathrm{D}^2 f(t) = \frac{(1-q^{-1})^2}{h^2} f(Kh).$$

The generalization to any order (real or complex) is immediate

$$\mathrm{D}^n f(t) = \frac{(1-q^{-1})^n}{h^n} f(Kh),$$

where n is real or complex.

Developing $(1-q^{-1})^n$ from the Newton binomial formulae gives

$$\mathrm{D}^n f(t) = \frac{1}{h^n}\left(\sum_{k=0}^{\infty}(-1)^k \frac{n(n-1)\cdots(n-k+1)}{k!} q^{-k}\right) f(Kh).$$

Since $q^{-k} f(Kh) = f((K-k)h) = f(t-kh)$, an other representation of the fractional derivative is

$$\mathrm{D}^n f(t) = \frac{1}{h^n}\sum_{k=0}^{\infty}(-1)^k \frac{n(n-1)\cdots(n-k+1)}{k!} f(t-kh).$$

Considering $f(t) = 0$ for $t < 0$, one has $f(t-kh) = 0$ for $t - kh < 0$. Thereby the infinite sum in the previous equation reduced to

$$\mathrm{D}^n f(t) = \frac{1}{h^n}\sum_{k=0}^{K}(-1)^k \frac{n(n-1)\cdots(n-k+1)}{k!} f(t-kh).$$

Appendix B. Model output sensitivity with respect to *n*

The output sensitivity with respect to n is defined by

$$\frac{\partial \hat{T}(t,\hat{\boldsymbol{\theta}})}{\partial n} = \mathbf{L}^{-1}\left(\sum_{l=1}^{N} -\frac{A_l s^n \ln(s)}{(s^n - \lambda_l)^2}\right) * \varphi(t) = \sum_{l=1}^{N} -A_l \mathbf{L}^{-1}\left[G_l\left(s,\hat{\boldsymbol{\theta}}\right)\right] * \varphi(t).$$

Using the Euler approximation, $s = (1-z^{-1})/h$, the Z transform corresponding to $G_l(s,\hat{\boldsymbol{\theta}})$ is

$$G_l(z,\hat{\boldsymbol{\theta}}) = \frac{((1-z^{-1})/h)^n \ln((1-z^{-1})/h)}{((1-z^{-1})/h)^{2n} - 2\lambda_l((1-z^{-1})/h)^n + \lambda_l^2}.$$

The Newton binomial allows the non-integer derivatives to be expressed by integer series expansion

$$G_l\left(z,\hat{\boldsymbol{\theta}}\right) = \frac{\left[\sum_{k=0}^{\infty}\frac{(-1)^k}{h^n}\binom{n}{k} z^{-k}\right][-\ln(h) + \ln(1-z^{-1})]}{\sum_{k=0}^{\infty}\frac{(-1)^k}{h^{2n}}\binom{2n}{k} z^{-k} - 2\lambda_l \sum_{k=0}^{\infty}\frac{(-1)^k}{h^n}\binom{n}{k} z^{-k} + \lambda_l^2}.$$

Integer series expansion of $\ln(1-z^{-1})$ gives then

$$G_l\left(z,\hat{\boldsymbol{\theta}}\right) = \frac{\left[\sum_{k=0}^{\infty}\frac{(-1)^k}{h^n}\binom{n}{k} z^{-k}\right]\left[-\ln(h) - \sum_{k=1}^{\infty}\frac{z^{-k}}{h}\right]}{\sum_{k=0}^{\infty}\left[\frac{1}{h^{2n}}\binom{2n}{k} - \frac{2\lambda_l}{h^n}\binom{n}{k}\right](-1)^k z^{-k} + \lambda_l^2}.$$

Finally, assuming that the system is relaxed at $t = 0$, a recursive equation system can be established

$$\left[\sum_{k=0}^{Kh}\left[\frac{1}{h^{2n}}\binom{2n}{k} - \frac{2\lambda_1}{h^n}\binom{n}{k}\right](-1)^k q^{-k} + \lambda_1^2\right]\frac{\partial y_1\left(Kh,\hat{\boldsymbol{\theta}}\right)}{\partial n} = \left[\sum_{k=0}^{Kh}\frac{(-1)^k}{h^n}\binom{n}{k} q^{-k}\right]\left[-\ln(h) - \sum_{k=1}^{Kh}\frac{q^{-k}}{h}\right] u(Kh)$$

$$\vdots$$

$$\left[\sum_{k=0}^{Kh}\left[\frac{1}{h^{2n}}\binom{2n}{k} - \frac{2\lambda_N}{h^n}\binom{n}{k}\right](-1)^k q^{-k} + \lambda_N^2\right]\frac{\partial y_N\left(Kh,\hat{\boldsymbol{\theta}}\right)}{\partial n} = \left[\sum_{k=0}^{Kh}\frac{(-1)^k}{h^n}\binom{n}{k} q^{-k}\right]\left[-\ln(h) - \sum_{k=1}^{Kh}\frac{q^{-k}}{h}\right] u(Kh)$$

$$\times \frac{\partial y\left(Kh,\hat{\boldsymbol{\theta}}\right)}{\partial n} = \sum_{l=1}^{N} A_l \frac{\partial y_l\left(Kh,\hat{\boldsymbol{\theta}}\right)}{\partial n},$$

where q^{-1} is the backward shift operator (unit time delay operator).

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